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A two-dimensional inverse problem of geometrical optics

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Abstract

In the framework of geometrical optics we consider the inverse problem consisting of obtaining refractive indexes $n = n(x, y)$ of a two-dimensional transparent heterogeneous isotropic (dispersive or not) medium from a known (observed or given) family $f(x, y) = c_0$ of planar light rays of a definite colour. We establish a first-order linear partial differential equation relating the assigned family of light rays with all possible refractive indexes compatible with this family. Using this equation we derive certain criteria to check whether a given family of rays can be traced in the presence of a refractive index, which we assume in advance to be either radial or homogeneous of any degree m. We give appropriate examples for the two special cases and also an example for the general case.

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1. Introduction

It is known that the mathematical treatment of the propagation of light makes use of two theories: the wave theory of light (physical optics) and the theory of light rays (geometrical optics). Both theories seem to be fundamentally different and can be developed independently of each other. Actually, however, they are connected. Both points of view are needed, even in problems of practical optical design [1–3].

The most general medium where light propagates is heterogeneous (or inhomogeneous) and anisotropic, but we shall confine our attention to the propagation of light in transparent media which are heterogeneous and isotropic, dispersive or not [4, 5]. This is, for instance, the case of the earth's atmosphere [1, 6], or of the lenses of some optical instruments [2, 10, 11]. We shall not consider absorbing media or non-isotropic media, such as metals or crystals.

In an heterogeneous isotropic medium the optical properties can be characterized by the refractive index of the medium

$$n = n(x, y, z) = \frac{c}{v}, \tag{1}$$

where x, y, z are rectangular Cartesian coordinates, c the velocity of light *in vacuo* and $v(x, y, z)$ the velocity of propagation of light at each point in the medium. It is assumed that, in each medium, the function $n(x, y, z)$ is continuous and possesses continuous partial derivatives.

From now on we shall adopt as $c = 1$ and we shall accept as a basis of our study the well-known (variational) Fermat's Principle: in transparent media the actual ray along which light travels from P_0 to P_1 has a stationary optical length when compared with adjacent curves joining P_0 to P_1, that is

$$\delta \int_{P_0}^{P_1} n \, \mathrm{d}s = 0, \tag{2}$$

where $\mathrm{d}s$ is the line element of the curves [1–3].

The direct problem of geometrical optics consists in finding the paths of light rays in a given transparent medium with known refractive index $n = n(x, y, z)$. The pertinent system of *ordinary* differential equations of the light rays can be found, of course, from (2).

In this paper we consider, with reference to an inertial Cartesian orthogonal system $Oxyz$, a two-dimensional transparent heterogeneous isotropic medium, the pertinent refractive index n of which depends only on the two variables x, y. In other words, we assume that the light rays lie in planes perpendicular to the z-axis of the system and the refractive index $n(x, y)$ is constant along each straight line parallel to the z-axis. So, it is not restrictive to study families of light rays in the Oxy plane.

In general the function n depends also on the colour of the light. In the present study we assume that all light rays, constituting the monoparametric family (3) below, are of the same colour. In other words we consider functions $n = n(x, y; \omega)$ with the cyclic frequency ω as a parameter common for all members of the family.

We look at the Fermat's principle from the point of view of the inverse problem, that is: given a family of rays to find compatible to it refractive indexes. More precisely we can state our version of the inverse problem of geometrical optics as follows:

Given a monoparametric family of monochromatic light rays in a transparent heterogeneous isotropic medium, $f(x, y) = c_0$ (c_0 being constant along each ray but varying from ray to ray), we want to find all the indexes of refraction $n(x, y)$ allowing for the creation of the given family. (For an account of inverse problems in optics see [7, 8].)

We prove that the requested functions $n = n(x, y)$ are solutions of a first-order linear partial differential equation (equation (12) below) which is our main result. It relates families of light rays and 'compatible' refractive indexes generating a given in advance or observed family of light rays. We shall call this equation *refractive index equation* (RIE).

In section 2 we derive the RIE from Fermat's principle, reasoning from the inverse point of view. In order to ease the mathematics involved, we seek refractive indexes $n(x, y)$ of one of the following two types: (i) radial: $n = n(r)$, $r = (x^2+y^2)^{\frac{1}{2}}$ (section 3), (ii) homogeneous of any degree m, that is of the form $n(x, y) = x^m R\left(\frac{y}{x}\right)$ with R arbitrary function of its argument (section 4). For each of the above two cases we shall show that, in general, the refractive index can be found uniquely (apart from a multiplicative constant), provided that the given family of rays or, more conveniently, the 'slope function' $\gamma(x, y)$, of the orthogonal trajectories of the family, defined below by (9), satisfies certain conditions.

Examples are presented for all cases and subcases studied in sections 3 and 4. There exist, of course, families of rays produced by refractive indexes not belonging to one of the above types, which are non-radial and non-homogeneous. An example originating from the literature is presented in section 5. The general comments of section 6 mostly refer to the physical justification of this work.

2. Partial differential equation for the refractive indexes

Suppose that we are given the equation

$$f(x, y) = c_0 \tag{3}$$

of a monoparametric family of light rays (normal congruence of curves) in a transparent isotropic heterogeneous medium (c_0 is constant on each ray, but varies from ray to ray). We shall find a PDE in $n = n(x, y)$ whose solutions give all the refractive indexes $n(x, y)$ compatible with the given family of rays (3).

We start with Fermat's principle (2). The line element ds, in Cartesian orthogonal coordinates, is

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} \tag{4}$$

so equation (2) reads

$$\delta \int_{x_0}^{x_1} n(x, y)\sqrt{1 + y'^2}\,\mathrm{d}x = 0. \tag{5}$$

The variational equation (5) is equivalent to Euler's ordinary differential equation

$$n_y\sqrt{1 + y'^2} - \frac{\mathrm{d}}{\mathrm{d}x}\left[n\frac{y'}{\sqrt{1 + y'^2}}\right] = 0 \tag{6}$$

where the subscript denotes partial derivative of the function n, and the prime denotes derivative with respect the variable x. By straightforward calculations we obtain

$$y'n_x - n_y + \frac{y''}{1 + y'^2}n = 0. \tag{7}$$

This differential equation of the light rays, as far as we know, has been interpreted, up to now, only from the point of view of the direct problem of geometrical optics, that is: given the refractive index $n(x, y)$ of the medium, to solve equation (7) as ordinary nonlinear differential equation of the second order in the unknown function $y = y(x)$.

We want now to transform the equation (7) to make it suitable for inverse problem considerations (that is given monoparametric families of rays to find compatible refractive indexes). To this end we proceed as follows: differentiating the equation of the family of rays (3) with respect to x, we obtain

$$y' = -\frac{f_x}{f_y}. \tag{8}$$

Introducing the slope function of the orthogonal trajectories of the family (3) (that is the traces in the plane Oxy of the *wave fronts* associated with the given family of rays)

$$\gamma(x, y) = \frac{f_y}{f_x}, \tag{9}$$

we write y' and y'' as functions of γ

$$y' = -\frac{1}{\gamma}, \qquad y'' = \frac{\Gamma}{\gamma^3} \tag{10}$$

where

$$\Gamma = \gamma\gamma_x - \gamma_y. \tag{11}$$

It is important to note that to each function $f(x, y)$ there corresponds one function $\gamma(x, y)$ and, vice versa, to each $\gamma(x, y)$ there corresponds one family (3).

Substituting (10) and (11) in (7) and considering n_x and n_y as partial derivatives of the unknown function $n(x, y)$, we obtain

$$\frac{\partial n}{\partial x} + \gamma(x, y)\frac{\partial n}{\partial y} = \Omega(x, y)n \tag{12}$$

with

$$\Omega(x, y) = \frac{\Gamma}{1+\gamma^2}. \tag{13}$$

Equation (12) is a linear partial differential equation of the first order in the unknown function $n = n(x, y)$ whose solutions provide all possible refractive indexes of the preassigned transparent medium capable to generate a given (or observed) family of light rays.

Comment. The partial differential equation (12) of the refraction indexes (RIE) can be derived also from the known vectorial equation of light rays

$$\kappa = \vec{\nu} \cdot \mathrm{grad}(\log n) \tag{14}$$

where κ is the curvature at a generic point of the light rays and $\vec{\nu}$ the normal unity vector (see formula (14) in section 3.2 of the treatise *Principles of Optics* by Born and Wolf [1]). We can transform this formula, from the inverse point of view, taking into account that

$$\kappa = \Gamma(1+\gamma^2)^{-\frac{3}{2}}, \qquad \nu_x = (1+\gamma^2)^{-\frac{1}{2}}, \qquad \nu_y = \gamma(1+\gamma^2)^{-\frac{1}{2}}. \tag{15}$$

If we insert (15) in (14) we obtain, by straightforward calculations, the partial differential equation (12). As seen by the first of equations (15), $\Gamma = 0$ is associated with a family of straight lines.

3. Radial refractive indexes

In this section, instead of $f(x, y) = c_0$, we consider the family (3) in polar coordinates r, θ, that is

$$f(r, \theta) = c_0 \tag{16}$$

and, instead of the slope function (9), we introduce the new function

$$\delta(r, \theta) = \frac{f_\theta}{f_r}. \tag{17}$$

Here again there exists an one-to-one correspondence of monoparametric families (16) and slope functions (17).

We propose the following problem: given the family of curves (16), to find all the refractive indexes which depend only on the distance r from a fixed point O (radially symmetric refractive indexes) $n = n(r)$, $r = (x^2+y^2)^{\frac{1}{2}}$ and which are compatible with the family (16) of light rays.

We take advantage of the special form of the refractive index and we rewrite equation (12) as follows:

$$rn_r + \frac{\delta}{r}n_\theta + \left(1 + \frac{\Psi}{\delta^2+r^2}\right)n = 0 \tag{18}$$

where

$$\Psi = \delta^2 + r(\delta_\theta - \delta\delta_r) \tag{19}$$

and $n = n(r, \theta)$ is to be found.

Let us consider the following two cases.

3.1. $n = n(r)$, any given $\delta = \delta(r, \theta)$

To have a solution of (18) of this form, we must have necessarily $\left(\frac{\Psi}{\delta^2+r^2}\right)_\theta = 0$ and this leads to the condition:

$$(r^2 + \delta^2)(\delta\delta_{r\theta} - \delta_{\theta\theta}) + [(r^2 - \delta^2)\delta_r + 2\delta\delta_\theta - 2r\delta]\delta_\theta = 0. \qquad (20)$$

So we arrive at:

Proposition 1. *The slope functions (17) of all families (16) compatible with radial refractive indexes $n = n(r)$ satisfy the differential condition (20).*

Example 1. It can be checked (e.g. by *Mathematica*) that the family of real branches of conics

$$f(r, \theta) = -r\cos\theta + \sqrt{r^2\cos^2\theta - 4r + 4} = c_0 \qquad (21)$$

satisfies the condition (20). Therefore there exist solutions

$$n = n(r) = n_0\sqrt{\frac{|r-2|}{r}}$$

with n_0 an arbitrary constant.

3.2. $n = n(r)$ and $f(r, \theta) = rg(\theta)$

The family consists of *geometrically similar* light rays and (17) gives

$$\delta = r\Delta(\theta) \qquad (22)$$

where

$$\Delta = \frac{g'}{g}, \qquad (23)$$

prime denoting derivative with respect to θ.

In this case the condition (20) becomes

$$(1 + \Delta^2)\Delta'' = 2\Delta(\Delta')^2 \qquad (24)$$

and the general solution of (24) is

$$\Delta = \tan(z_0\theta + z_1) \qquad (25)$$

with $z_0 \neq 0$, z_1 arbitrary constants. Inserting (22) with Δ given by (25) into (18), we obtain the refractive index

$$n(r) = n_0 r^{-(1+z_0)}. \qquad (26)$$

Finally, integrating (23) with Δ given by (25) we obtain

Proposition 2. *The families of curves (sinusoidal spirals)*

$$f(r, \theta) = r^{-z_0}\cos(z_0\theta + z_1) = c_0 \qquad (27)$$

are compatible with the refractive index (26).

Remark. If we put $z_0 = -\frac{1}{2}$, $z_1 = 0$ the family of light rays (27) becomes $r\cos^2\left(\frac{\theta}{2}\right) = \text{const.}$, that is a family of parabolas compatible with the refractive index $n(r) = \frac{n_0}{\sqrt{r}}$. This index of refraction is, for the earth's atmosphere, in agreement with the known *Simpson's hypothesis* (see Smart [6], p 72)

$$\left(\frac{n}{n_0}\right)^{\mu+1} = \frac{r_0}{r},$$

where n_0 and r_0 are respectively the refractive index and the radius at the point of observation (with $\mu = 1$). This formula was proposed also by the French astronomer P Bouguer.

4. Homogeneous refractive indexes

In this section we consider families of light rays (3) which we want to be traced in a medium with homogeneous refractive index $n(x, y) = x^m R\left(\frac{y}{x}\right)$, with degree of homogeneity m. The question is if, for a preassigned family of rays, such refractive indexes do exist.

We distinguish two cases:

4.1. $n =$ homogeneous of degree m, γ is not homogeneous of degree zero

We have

$$n(x, y) = x^m R(z), \qquad z = \frac{y}{x} \tag{28}$$

and equation (12) becomes

$$mR - zR' = x\Omega R - \gamma R' \tag{29}$$

where the prime denotes derivative with respect to z and where Ω is given by (13).

On the other hand, the above assumption regarding $\gamma(x, y)$ means that

$$x\gamma_x + y\gamma_y \neq 0. \tag{30}$$

The right-hand side of equation (29) must be a function of z (i.e. homogeneous in x, y of degree zero). Therefore

$$x(x\Omega R - \gamma R')_x + y(x\Omega R - \gamma R')_y = 0 \tag{31}$$

and this leads to

$$\frac{R'}{R} = \rho \tag{32}$$

where

$$\rho = \frac{x(x\Omega_x + y\Omega_y + \Omega)}{x\gamma_x + y\gamma_y}. \tag{33}$$

But $\rho(x, y)$ must be homogeneous of degree -1, i.e.

$$x\rho_x + y\rho_y + \rho = 0$$

and this leads to the condition for the given families

$$\begin{aligned}(x\gamma_x + y\gamma_y)&[x^2\Omega_{xx} + 2xy\Omega_{xy} + y^2\Omega_{yy} + 2(x\Omega_x + y\Omega_y)]\\ &= (x^2\gamma_{xx} + 2xy\gamma_{xy} + y^2\gamma_{yy})(x\Omega_x + y\Omega_y + \Omega).\end{aligned} \tag{34}$$

Therefore we can state the following:

Proposition 3. *Any family of rays $\gamma(x, y)$ satisfying (30) and (34) is compatible with all homogeneous refractive indexes (28) found from (32), after solving it for R by quadratures.*

Example 2. The functions

$$\gamma = \sqrt{ax^m - 1} \tag{35}$$

satisfy the condition (34). From (33) we have $\rho = 0$, therefore, from (32) and (28) we obtain

$$n = n_0 x^m. \tag{36}$$

4.2. n homogeneous of degree m, γ homogeneous of zero degree

Then equation (12), (or equation (29) with $\gamma = \gamma(z)$, $\Omega = -\frac{\gamma'(1+z\gamma)}{x(1+\gamma^2)}$) becomes

$$\frac{R'}{R} = \tau \tag{37}$$

where

$$\tau = \frac{1}{z-\gamma}\left[m + \frac{(1+z\gamma)\gamma'}{1+\gamma^2}\right]. \tag{38}$$

So we state

Proposition 4. *For any m and any $\gamma = \gamma(z) \neq z$ the homogeneous refractive indexes (28) compatible with the given $\gamma(z)$ are found by quadratures from (37).*

Example 3. For $\gamma = -z$, (i.e. for the family of hyperbolas $x^2 - y^2 = c_0$), it is $\tau = \frac{1}{2z}\left(m - \frac{1-z^2}{1+z^2}\right)$ and, from (37), we find

$$R = R_0 z^{\frac{m-1}{2}}\sqrt{1+z^2}, \tag{39}$$

so the pertinent refractive index is

$$n(x,y) = R_0 x^m \left(\frac{y}{x}\right)^{\frac{m-1}{2}} \sqrt{1+\left(\frac{y}{x}\right)^2}.$$

Remark 1. If we put in the previous formula $m = 1$ we find $n(x,y) = R_0\sqrt{x_2 + y^2} = R_0 r$ that is a homogeneous first degree refractive index proportional to r and compatible with the family of hyperbolas $x^2 - y^2 = c_0$. This could appear in a large layer of the atmosphere near the earth's surface when this surface becomes very hot by the Sun radiation. Then the refractive index increases from the surface of the earth to some altitude where it takes the maximum value before decreasing (mirage effect).

Remark 2. Consider the family of concentric circles in polar coordinates

$$r^2 = c_0. \tag{40}$$

In this case $\gamma = z$ and equation (18) of refraction indexes reads

$$nn_r + n = 0 \tag{41}$$

that is integrable by quadrature. Therefore, we conclude

Proposition 5. *All homogeneous refractive indexes producing as light rays the family of circles (40) are of the form*

$$n(r,\theta) = \frac{F(\theta)}{r}, \tag{42}$$

where $F(\theta)$ is an arbitrary positive function. They are all of degree $m = -1$.

Remark 3 (regarding families of straight lines)**.** If $\Gamma = \gamma\gamma_x - \gamma_y = 0$, then, from (12), $\gamma = -\frac{n_x}{n_y}$. Inserting this into $\gamma\gamma_x - \gamma_y = 0$ we obtain

$$\left({n_y}^2 - {n_x}^2\right)n_{xy} = n_x n_y (n_{yy} - n_{xx}). \tag{43}$$

This leads to

Proposition 6. *All refractive indexes satisfying (43) allow, among others, for families of straight lines rays given by $\gamma = -\frac{n_x}{n_y}$. Such are for instance refractive indexes of the form $n = n(x)$ or $n = n(y)$.*

5. General refractive indexes

We now apply equation (12) without imposing any limitations on the functions $n(x, y)$ and $f(x, y)$, and we comment on a practical example taken from the literature.

Fletcher *et al* [10] studied the distribution of the refractive index in a symmetrical cylindrical lens terminated by a circular planar face which has, as a centre, the origin O of the system of coordinates and which is perpendicular to the axis of symmetry Ox. All light rays parallel to Ox outside the lens, where $n_0 = 1$, are curved inside the lens and pass through the same focus F on Ox at a distance $OF = f^*$ from the planar face. The problem is essentially the same as a two-dimensional problem with plane layers parallel to Ox. Assuming that $n = n(r)$ ($r = |y|$ is the distance of any point of the ray from the axis Ox), they reduced the problem to two dimensions and found that the family of rays

$$x - \frac{1}{\alpha}\arcsin[c_0 \sinh(\alpha y)] = f^*, \tag{44}$$

is compatible with the refractive index

$$n = n_0 \operatorname{sech}(\alpha y) \tag{45}$$

($\alpha = \frac{\pi}{2f^*}$ = constant, n_0 = constant, c_0 is the parameter of the family).

We note that the parameter c_0 in (44) does not enter into (45). Solved for c_0, equation (44) reads

$$f(x, y) = \frac{\sin[\alpha(x - f^*)]}{\sinh(\alpha y)} = c_0. \tag{46}$$

Let us now free ourselves from the limitation $n = n(r) = n(y)$ and look for *all* refractive indexes compatible with the family (46). For this case the PDE (12) reads

$$n_x - \frac{\tan[\alpha(x - f^*)]}{\tanh(\alpha y)} n_y = n\alpha \tan[\alpha(x - f^*)]. \tag{47}$$

The general solution of (47) is

$$n = \frac{1}{\cosh(\alpha y)} \Phi\left(\frac{\cos[\alpha(x - f^*)]}{\cosh(\alpha y)}\right), \tag{48}$$

where Φ is an arbitrary function of its argument. For $\Phi = n_0$, equation (48) gives the known solution (45), established by Fletcher *et al*.

Remark. We observe that the focus $F = (f^*, 0)$ is a singular point for the first member of the family in the form (46), but not in the form (44). Of course if we take the limit of $\frac{\sin[\alpha(x-f^*)]}{\sinh(\alpha y)}$ with $(x, y) \to (f^*, 0)$ on each curve of the family (44) we find the pertinent value of the parameter c_0.

6. General comments

In this paper we studied in some detail the possibility of obtaining refractive indexes $n = n(x, y)$ of two variables under the assumption that a family of planar light rays (given or observed) is known. On the grounds of Fermat's principle, we derived what we called the refractive index equation (RIE)(12). This PDE in $n = n(x, y)$ is our basic result. Very recently Fermat's principle was applied by Marklund *et al* ([12]) and was discussed also by West ([13]) in optical models with a singularity in the centre. It was found that light trajectories are similar to those around a black hole in the sense that, beyond a critical radius, the light cannot escape but spirals into the singularity.

If no assumption is made regarding the type of the refractive index, the RIE (12) has as many solutions as an arbitrary function introduces. This totality of solutions was found by (48) in section 5 for the family given by (44) or, equivalently, by (46). Yet, in general, these solutions cannot be found because the pertinent subsidiary system of ODEs, corresponding to the PDE (12), is nonlinear.

In order to ease the mathematics and to reduce the multitudiness of possible solutions, we put certain restrictive assumptions regarding the form of the unknown refractive indexes (e.g. radial or homogeneous). These assumptions generally imply certain criteria in the form of differential conditions (e.g. the conditions (20) and (34)) which must be satisfied by the 'given' family of rays.

As regards the physical justification of the above assumptions, we know that optical media with spherical refractive index $n = n(r)$ are of considerable theoretical and practical interest as they represent, in a certain sense, perfect optical instruments (Luneburg [2], p 164). On the other hand, planar models are justified when $n = n(r)$ because light rays in three-dimensional media are lying on planes passing through the origin in the same manner as material particles move in central force fields. Examples are as follows.

(i) The optical medium characterized by the refractive index function $n(r) = n_0/[1+(r/a)^2]$, where n_0 and a are constants. This is an absolute instrument, known as the *Maxwell's fish-eye*, it has the interesting properties that the light rays are circles and the imaging is an *inversion* ([1], p 147 and [2], p 172).

(ii) The so-called *Luneburg lens* is an inhomogeneous sphere with the refractive index function $n(r) = \sqrt{2 - r^2} (0 \leqslant r \leqslant 1)$, [2], p 187. When immersed in a homogeneous medium of unity refractive index, it brings to a sharp focus, on the surface of the sphere, every incident pencil of parallel rays, and vice versa, all light rays entering the sphere from any fixed point P on its surface emerge from the sphere parallel to the diameter OP, [9–11]. Because of its wide angle scanning capabilities it has useful applications in microwave antenna design (see Rinehart [9]). Another practical application was considered by Fletcher *et al* in connection to some researches on the eyes of fishes [10].

(iii) The Earth's (as well as other planets') atmosphere is assumed to have $n = n(r)$ and, on this basis, astronomers apply corrections to the observed zenith distances z of stars. Not very close to the horizon, to a satisfactory approximation, these corrections are taken proportional to $\tan z$ (see Smart, [6] pp 60–2).

(iv) As is known to astrophysicists, the generally spherical shape of stars tends to produce spherically symmetrical refractive indexes. Einstein noted that gravity, as it curves the ray paths in the neighboruhood of large masses, causes a certain change in the index of refraction for the travelling electromagnetic wave ([14], pp 190–1).

In general, if the family of rays is preassigned, one is not expecting an affirmative answer to the question of the existence of a refractive index of the above types (radial or homogeneous). If, however, the pertinent criteria are fulfilled, the corresponding refractive index is usually found up to certain integration constants. In contrast, if no limiting assumption is made for the demanded function $n = n(x, y)$ and the 'given' family $f(x, y) = c_0$, one expects infinitely many refractive indexes to allow for the creation of this family. This was illustrated by the example of section 5.

The following comment refers to possible practical applications of the PDE (12). In order to use it profitably we need possess a monoparametric family of light rays (3). Apparently this is not feasible with data collected by observations. The best we can do with real measurements data is to obtain (by a curve-fitting procedure) analytical equations $F(x, y) = 0$ for a number of light rays and, out of these, to guess the form (3) of the family having these rays as members.

Finally we note that, taken into account the well-known Hamiltonian optico-mechanical analogy (see appendix II of [1]), the results of this paper for geometrical optics are in agreement with analogous results obtained recently in the framework of the inverse problem of particle dynamics (see [15, 16]).

This study may be extended to other types of refractive indexes in two variables $n = n(x, y)$ and also to the general case of three variables $n = n(x, y, z)$.

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